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The NP-Completeness Column: An Ongoing Guide

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This is the third edition of a quarterly column the purpose of which is to provide a continuing update to the list of problems (NP-complete and harder) presented by M. R. Garey and myself in our book ''Computers and Intractability: A Guide to the Theory of NP-Completeness,'' W. H. Freeman & Co., San Francisco, 1979 (hereinafter referred to as ''[G&J]''; previous columns will be referred to by their dates). A background equivalent to that provided by [G&J] is assumed. Readers having results they would like mentioned (NP-hardness, PSPACE-hardness, polynomial-time-solvability, etc.), or open problems they would like publicized, should send them to David S. Johnson, Room 2C-355, Bell Laboratories, Murray Hill, NJ 07974, including details, or at least sketches, of any new proofs (full papers are preferred). In the case of unpublished results, please state explicitly that you would like the results to be mentioned in the column. Comments and corrections are also welcome. For more details on the nature of the column and the form of desired submissions, see the December 1981 issue of this journal.

1. INTRODUCTION

The response to this column and my call for contributions has been gratifying. At the rate new results are arriving, it is unlikely that I shall soon run out of things to report. The backlog is, in fact, increasing, although there is fortunately some overlap: two results in this column each came from *three* independent sources. My thanks to all those who have sent contributions, and my apologies for the tardiness of my replies to queries. As I succeed in automating more of the clerical work involved with the column, some of these delays should be eliminated. Delays in the appearance of results in the column will, however, be unavoidable, given the quantity of material on hand.

Comments and corrections on earlier columns will have priority. Given that I am now typesetting the column myself (using the UNIX® system here at Bell Laboratories), publication delays have been minimized, and so, for example, responses to the current column can be reported in December, quite a good turnaround time for a scientific journal. On the other hand, new results will experience variable delays, depending on their subject matter. For the time being I shall continue my policy of organizing each edition of the column around a particular topic or theme. This month's topic is partitioning, covering, and packing, and the main body of the column will be devoted to problems of this type. A brief concluding section will provide an update on problems in [G&J] relevant to this month's topic, and on results discussed in the first two editions of the column.

2. PARTITIONING, COVERING, AND PACKING PROBLEMS

Problems of partitioning, covering, and packing are often closely related. In a partitioning problem, we are given an object and are asked for a collection of disjoint sub-objects whose union equals the given object; in a covering problem we are asked for a (not necessarily disjoint) collection whose union *contains* the given object; in a packing problem we are asked for a collection of disjoint objects whose union is *contained in* the given object.

An example of just how closely related these three problems can be is provided by the PARTITION INTO TRIANGLES problem [GT11] of [G&J]: ''Given a graph $G = (V,E)$, can its vertices be partitioned into three-element sets V_1, V_2, ..., such that each V_i induces a triangle (complete graph K_3) in G?'' The NP-completeness of this problem for graphs containing no complete graphs K_4 as subgraphs, proved in [39], immediately implies NP-completeness for the question of whether the vertices of a given graph G can be covered by K or fewer complete subgraphs of G, and for the question of whether K or more disjoint triangles can be packed into a given graph G.

The above problem provides an apt introduction, as problems of partitioning into ''triangles'' will be encountered twice more before we are through with this month's discussion. We present the first such problem immediately. It is concerned with partitioning the edges, rather than the vertices, of a graph, and requires slightly more delicate arguments.

[1] EDGE-PARTITION INTO TRIANGLES

INSTANCE: Graph $G = (V,E)$.

QUESTION: Is there a partition $E = E_1 \cup \cdots \cup E_k$ into disjoint sets such that each E_i is isomorphic to the complete graph K_3?

Reference. Holyer [18]. Transformation from 3SAT.

Comment. Remains NP-complete even if G contains no complete subgraph of size greater than 3. The problem is also NP-complete if K_3 is replaced by K_n for any $n > 3$. The result for K_3 implies NP-completeness for corresponding covering and packing problems, just as did (vertex) PARTITION INTO TRIANGLES, although note that the corresponding covering problem was already listed in [G&J], under the name of COVERING BY CLIQUES [GT17].

Our next problem is a variant on the CHROMATIC NUMBER problem [GT4]. Recall that CHROMATIC NUMBER asks whether a given graph G can have its vertices partitioned into K or fewer independent sets (and hence is in a sense dual to PARTITION INTO TRIANGLES, which asks for a partition into cliques).

[2] DISTANCE-d CHROMATIC NUMBER

INSTANCE: Graph $G = (V,E)$, positive integer $K \le |V|$.

QUESTION: Is there a partition $V = V_1 \cup \cdots \cup V_K$ such that if u and v are distinct elements of some set V_i in the partition, then there is no path from u to v in G of length d or less?

Reference. McCormick [29]. Transformation from 3SAT.

Comment. NP-complete for any fixed $d \ge 1$. The case for $d = 1$ is ordinary CHROMATIC NUMBER; the cited reference proves NP-completeness for all fixed $d \ge 2$. The problem for $d = 2$ models certain partitioning questions that arise in procedures for computing approximations to sparse Hessian matrices [13,29,36]: Given an $n \times n$ matrix A of 0's and 1's, one wishes to partition the columns into a minimum number of sets so that no two columns in the same set have a 1 in the same row. One obtains the $d = 2$ problem if one takes A to be the adjacency matrix for a graph with 1's along the main diagonal, and [29] investigates approximation algorithms for this case. A related problem which remains open asks for a partition of the columns of A into a minimum number of sets such that for each i, j with $a_{ij} = 1$, the set that contains column j contains no other column with a 1 in row i. See [29] for further details.

The next problem is in a sense a geometric version of the last.

[3] DISTANCE-d PARTITION OF POINTS IN THE PLANE

INSTANCE: Finite set $P \subset Z \times Z$ of integer-coordinate points in the plane, positive integers d and K.

QUESTION: Is there a partition $P = P_1 \cup \cdots \cup P_K$ such that if u and v are distinct points in some set P_i of the partition, then the Euclidean distance $d(u,v) > d$?

Reference. Burr [5]. Transformation from PLANAR GRAPH 3-COLORABILITY.

Comment. Remains NP-complete for fixed $K = 3$, even if the Euclidean metric is replaced by the rectilinear one. The dual problem, in which $d(u,v)$ must be *less* than d when u and v are in the same set is also NP-complete, as can be seen by examining the NP-completeness constructions used for the GEOMETRIC COVERING BY CIRCLES problem, mentioned below. Both of these partitioning problems can be viewed as special cases of CHROMATIC NUMBER and hence are solvable in polynomial time for $K \le 2$.

[4] GEOMETRIC COVERING BY DISCS

INSTANCE: Set P of integer-coordinate points in the plane, positive integers d and K.

QUESTION: Can the points of P be covered by K discs of diameter d?

Reference. Proved independently by Fowler *et al.* [15], Masuyama *et al.* [28], and Supowit [41]. Transformation from PLANAR 3SAT.

Comment. Remains NP-complete if the centers of the discs are required to be points from our set P [28], in which case the problem corresponds to the (Euclidean) GEOMETRIC CONNECTED DOMINATING SET [ND48] of [G&J], with the connectivity constraint removed. The variants on these problems in which discs are replaced by squares of a fixed size (aligned with the coordinate axes) are also NP-complete [15,28], with the square being to the rectilinear metric what the disc is to the Euclidean. The problem for squares was motivated by a practical question related to databases used in image processing [15]; the problem for discs relates to the determination of optimal locations for fire stations or other emergency services [28] (when the research is funded by civilian agencies, that is; if the research is funded by the military the problem becomes one of ‘‘optimal bomb targeting’’).

So far we have considered problems in which the partitions or covers were made up of a large number of sets. The next five problems concern divisions into just *two* parts.

[5] INTEGRITY PRESERVING CUT

INSTANCE: Graph $G = (V,E)$, disjoint subsets U_1, $U_2 \subset V$.

QUESTION: Is there a partition of V into disjoint sets V_1 and V_2 such that, for each $i \in \{1,2\}$, $U_i \subseteq V_i$ and the subgraph of G induced by V_i is connected?

Reference. Luby [26]. Transformation from 3SAT.

Comment. This problem arose in conjunction with an attempt to generate the K best *minimal* $s-t$ cuts in an undirected graph (see KTH BEST s-t CUT, below).

However, one can also imagine a geopolitical application: How can one divide an ethnically polarized country into two new nations, each having a secure road network linking its main cities? The question thus arises, does the problem remain NP-complete when restricted to planar graphs? Note however that the result for general graphs may still be relevant. Whereas most countries were once at least topologically (if not topographically) equivalent to the plane, the presence of tunnels and overpasses in modern day road systems gives rise to the possibility of countries with arbitrarily large genus.

The next problem asks for connected parts with bounded size rather than specified contents (*some* restriction must be added to connectivity, as just asking for connected parts is, of course, no challenge).

[6] CUT INTO CONNECTED COMPONENTS OF BOUNDED SIZE

INSTANCE: Graph $G = (V,E)$, integer bound $K \geq |V|/2$.

QUESTION: Is there a partition of V into disjoint sets V_1 and V_2 such that $\max(|V_1|, |V_2|) \leq K$, and both V_1 and V_2 induce connected subgraphs of G?

Reference. Frieze [16]. Transformation from 3DM.

Comment. Remains NP-complete if G is bipartite; however is solvable in polynomial time if G is 2-connected (has no articulation points), for then the answer is always ‘‘yes’’. The problem for planar graphs with articulation points remains open. However, if weights are assigned to the vertices, and we ask for sets V_i whose total *weight* is bounded by K, the problem becomes NP-complete even for bipartite planar 2-connected graphs.

The next problem asks for parts with a certain *lack* of connectivity.

[7] CUT INTO ACYCLIC SUBGRAPHS

INSTANCE: Directed graph $G = (V,A)$.

QUESTION: Is there a partition of V into disjoint sets V_1 and V_2 such that both V_1 and V_2 induce acyclic subgraphs (ones containing no nontrivial strongly connected components)?

Reference. Gács [17]. Transformation from 3SAT.

Comment. This problem is equivalent to a matrix problem arising in the design of optimizing compilers: An $n \times n$ matrix A is *segmented* if for some k, $a_{ij} = 0$ for all $1 \leq j < i \leq k$ and $k < i < j \leq n$. The current question is equivalent to asking whether a matrix A is *segmentable*, i.e., whether there is a permutation matrix P such that $P^{-1}AP$ is segmented.

The next problem is analogous to problems like K^{th} BEST SPANNING TREE [ND9] in [G&J], in that it concerns a problem where finding the *best* object is easy, but telling how *many* objects meet a given threshold is hard.

[8] K^{th} BEST s-t CUT

INSTANCE: Graph $G = (V,E)$, distinct vertices $s,t \in V$, bounds $B,K \in Z^+$.

QUESTION: Are there distinct sets $C_1, \ldots, C_K \subseteq E$ such that for each i, $1 \leq i \leq K$, $|C_i| \leq B$ and every path from s to t in G must use an edge from C_i (and hence when the edges in C_i are removed from G, s and t will be in separate connected components)?

Reference. Provan and Ball [37]. Transformation from the #P-complete problem of counting the number of vertex covers of a graph.

Comment. Not known to be in NP. When $K = 1$, this is the famous ''max flow/min cut'' problem, and can be solved in polynomial time. Moreover, even when K is arbitrary, the problem can be solved in pseudo-polynomial time (i.e., time polynomial in K and the size of the graph), using a general purpose technique due to Lawler [21] for generating the K best solutions to a 0-1 programming problem. It is not known whether the K best *minimal* $s-t$ cuts can be generated in pseudo-polynomial time, although *all* minimal $s-t$ cuts can be generated in time polynomial in their number and the size of the graph (e.g., see [42]). The difficulty is that their number may be an exponential function of the size of the graph.

The actual result in [37] is that the problem of determining the number of optimal (minimum sized) $s-t$ cuts is #P-complete; I have restated the problem in decision problem form merely for consistency. The related problem in which G is directed is also #P-complete, as is the ''directed network cut'' problem, where the sets C_i are only required to cut all directed paths from s to *some* vertex, the identity of the vertex depending only on C_i. The *undirected* network cut enumeration problem can be solved in polynomial time [1], and the directed network cut problem can be so solved when restricted to planar or acyclic graphs [1]. The restriction to planar graphs also makes both $s-t$ cut enumeration problems solvable in polynomial time [1], although the restriction to acyclic graphs leaves the directed $s-t$ cut problem #P-complete [37]. Provan and Ball have also shown that a variety of other enumeration problems are #P-complete in [37], among them the problems of counting the number of independent sets (or maximum independent sets) in a bipartite graph and the number of antichains (maximum antichains) in a partial order. Note that in these cases too, one can find an optimum set of the required form in polynomial time, and the K best in pseudo-polynomial time.

We now turn from problems with numbers as output to one with numbers as input. Many number problems can be proved NP-complete by straightforward transformations from such basic problems as PARTITION and 3-PARTITION. In general I will not be listing such results in this column, as I assume that most readers are capable of coming up with the required proof on their own, should the need arise. The next problem, however, although it looks superficially like PARTITION, is actually quite a bit more subtle.

[9] WEAK PARTITION

INSTANCE: A finite set A and a *size* $s(a) \in Z^+$ for each $a \in A$.

QUESTION: Are there disjoint non-empty subsets $A_1, A_2 \subset A$ such that

$$\sum_{a \in A_1} s(a) = \sum_{a \in A_2} s(a) \ ?$$

Reference. Shamir [40] (independently rediscovered by van Emde Boas [43] and Rubin [38]). Transformation from PARTITION.

Comment. This problem can be restated in terms of linear equations: Is there an integer-valued solution to the equation $\sum_{i=1}^{|A|} x_i \cdot s(a_i) = 0$, in which $|x_i| \le 1$ for all i and at least one x_i is non-zero? (PARTITION corresponds to the case where all x_i must be ± 1). This problem remains NP-complete even if we allow solutions with $|x_i|$ as large as K, for any fixed K [42]. Applications of WEAK PARTITION to cryptography are discussed in [40], whereas [42] considers applications to problems of computing ''short'' vectors in lattices.

Our remaining problems concern the partitioning, covering, and packing of figures in the plane. Given an arbitrary polygon in the plane, there are many reasons we might wish to break it up into subfigures. In some cases this may be an aid to pattern recognition; in others, an algorithm that only works for a restricted class of figures, such as convex polygons, may be applicable to an arbitrary polygon if we first decompose that polygon into subfigures belonging to the restricted class. Such a decomposition may also serve as a convenient method for storing a representation of the figure in useful form, as in the description of VLSI designs. Our first problem concerns arbitrary polygons, and is our promised third ''partition into triangles'' problem.

[10] GEOMETRIC PARTITION INTO TRIANGLES

INSTANCE: A polygon P (not necessarily simply connected), described by a sequence of pairs (p_1, p_2) of integer-coordinate points in the plane, each pair representing the endpoints of a line segment on the boundary of P, and a positive integer K.

QUESTION: Can P be partitioned into a set of K or fewer disjoint triangles, all of whose corners are vertices of P (i.e., with no ‘‘Steiner points’’ allowed)?

Reference. Lingas [22]. Transformation from PLANAR 3SAT.

Comment. The complexity of this problem depends on the possibillity that a vertex on the boundary of a hole in P may be collinear with two points on the exterior boundary or the boundary of another hole. If there is no set of more than two collinear points, then the minimum number of triangles is precisely determined by Euler’s formula, and the desired triangulation can be constructed easily. If P is simply connected (and hence *has* no holes), then the problem can be solved in polynomial time even in the presence of collinearities, using dynamic programming [35].

If we allow our decomposition to have Steiner points, i.e., vertices which are not vertices of the original polygon, then polygon partitioning problems are potentially more complex, as there is a possibility that they may not even be in NP. Hence what is surprising about the next two problems may well be their polynomial time solvable subcases, rather than the NP-hardness of the problems themselves.

[11] PARTITION INTO CONVEX REGIONS

INSTANCE: Same as in previous problem.

QUESTION: Can P be partitioned into K or fewer disjoint convex polygons, where a polygon is *convex* if any straight line segment joining two of its points lies entirely within the polygon and its boundary?

Reference. Lingas [22]. Transformation from PLANAR 3SAT.

Comment. Not known to be in NP. A polynomial time algorithm has been claimed for the case where P has no holes [7,9,10], although the result has not yet been verified by referees. If one generalizes this problem to three dimensions, and asks for the partition of a polyhedron into K or fewer convex polyhedra, the problem becomes NP-hard even if P has no holes, as a straightforward corollary to the current result [22]. However, there does exist an algorithm that will construct a partition of a 3-dimensional polytope into $O(n^2)$ convex parts — and there exist P for which $\Omega(n^2)$ is the best possible [8]. The variant of the 2-dimensional problem in which the regions must all be triangles, possibly using Steiner points, is also NP-hard if P is allowed to have holes [22]. The variant in which P is a *rectilinear* polygon (all its edges are either vertical or horizontal line segments), and one asks for a partition into rectangular regions, is solvable in polynomial time by a matching technique [25], even in the presence of holes.

However, if one allows degenerate holes, i.e., isolated points inside of P which are not allowed to be interior to any of the rectangles in the partition, then even this rectilinear version of the problem becomes NP-hard (although it is at least in NP, and hence ‘‘only’’ NP-complete [22]).

The partitioning of rectilinear polygons, mentioned above, has potential applications to VLSI design, where commonly used design rules and practices result in rectilinear designs. Other applications might be to partitioning a floor of a building into rooms. If all rooms are to be rectangular, if there are interior airshafts that constrain the floorplan, and if constructing the walls of the rooms is the main source of expense, one might come up with the following problem.

[12] MINIMUM PERIMETER PARTITION INTO RECTANGLES

INSTANCE: Rectilinear, not necessarily simply connected polygon P, positive integer K.

QUESTION: Can P be partitioned into disjoint rectangles R_1, R_2, ..., in such a way that the sum of the perimeters of the rectangles R_i is K or less?

Reference. Lingas, Pinter, Rivest, and Shamir [24]. Transformation from PLANAR 3SAT.

Comment. Solvable in polynomial time when restricted to rectilinear polygons without holes [24]. If degenerate holes (points) are allowed, problem is NP-complete if these may be *corectilinear* (two or more may occur on the same horizontal or vertical line). Still open is the case where the only holes allowed are non-corectilinear points. Approximation algorithms are discussed in [23].

Our final two problems concern the covering and packing of polygons.

[13] COVERING BY CONVEX POLYGONS

INSTANCE: Same as for GEOMETRIC PARTITION INTO TRIANGLES.

QUESTION: Are there convex polygons P_i, $1 \le i \le K$, whose union (as planar figures) is P?

Reference. O’Rourke and Supowit [32]. Transformation from PLANAR 3SAT.

Comment. Not known to be in NP. Variants in which the P_i must all be ‘‘star-shaped’’ or must all be ‘‘spirals’’ are also NP-complete. Once again, the proofs require that the original polygon contain holes, and the problem is open if restricted to simply connected figures. If P is rectilinear (but not necessarily simply connected), and we ask for a cover by rectangles (with horizontal and vertical sides), we get the NP-complete RECTILINEAR PICTURE COMPRESSION problem [SR25] of [G&J]. This latter problem can be solved in polynomial time if restricted to ''rectilinearly convex'' polygons P, where a polygon is convex in this sense if any horizontal or vertical line segment connecting two of its points lies entirely within the polygon and its boundary [6]. The rectilinear problem remains open for the case of arbitrary simply connected rectilinear polygons. Note that the NP-completeness of RECTILINEAR PICTURE COMPRESSION does not immediately imply that of COVERING BY CONVEX POLYGONS, since there exist examples of rectilinear polygons whose minimum convex cover involves polygons whose edges are not all horizontal and vertical line segments [31].

[14] PACKING WITH SQUARES

INSTANCE: Rectilinear, not necessarily simply connected polygon P, positive integers d and K.

QUESTION: Can K disjoint $d \times d$ squares be placed inside P with their sides parallel to the axes?

Reference. Fowler *et al.* [15]. Transformation from PLANAR 3SAT.

Comment. Remains NP-complete even if d is fixed at 2 [3]. However, if the 2×2 square is replaced by a 1×2 rectangle, the problem is solvable in polynomial time by bipartite matching. An application of this problem, suggested in [3] is to increasing yields in VLSI chip manufacture. In oversimplified terms, suppose we are manufacturing wafers filled with a rectilinear array of what are to become 64K RAM chips. With a little extra effort in the design phase, it could be arranged that any 2×2 subarray of these chips could become a 256K RAM. If a particular wafer is produced with some of its 64K RAM's defective, we then would want to find the maximum number of disjoint 2×2 subarrays that could be cut out of the functioning parts of the wafer, and these functioning parts together form what is essentially a union of rectilinear polygons. (The problem of packing such an area with identical discs is also NP-complete [15], with the obvious consequences for our attempts to consruct πr^2K RAM's).

We conclude this section with an ''open problem of the month'' that is, appropriately enough, a covering problem. In [G&J], it was mentioned in the comments to DECODING OF LINEAR CODES [MS7], and also as a problem that, like the then-open CHROMATIC INDEX problem, involved issues of ''parity''. Now that the CHROMATIC INDEX problem is known to be NP-complete, as reported in the [Dec. 1981] issue, this other ''parity'' problem seems worth highlighting on its own, especially since there are new partial results to be reported, and the problem bears at least a superficial resemblence to the WEAK PARTITION problem discussed above.

[OPEN] EVEN COVER

INSTANCE: Collection C of subsets of a given finite set X, positive integer K.

QUESTION: Is there a non-empty subcollection $C' \subseteq C$ with $|C'| \le K$, such that each element of X is in an even number (possibly zero) of sets from C' ?

Comment. Corresponds to the problem of determining, for a given linear code, whether there exists a codeword of ''Hamming'' weight K or less [2]. If we require that $|C'| = K$, the problem becomes NP-complete [2], as was mentioned in [G&J]. The problem also becomes NP-complete if we require that $|C'| > K$, or that $|C'|$ lie between two given bounds, or that $|C'|$ be less than K and not be divisible by a specified integer $k \ge 2$ [30]. The problem is solvable in polynomial time if all $c \in C$ have size two or less [20]. It is also solvable in polynomial time if $K \ge |C|$, as it then reduces to simply solving a system of linear equations over GF[2]. However, if K is this large but we specify some even integer $k \ge 4$ and require that for each element of $x \in X$, $|\{c \in C' : x \in C'\}|$ be an even number *other* than k, the problem once more becomes NP-complete [20].

4. UPDATES

New special case results of interest have been proved concerning two problems of partitioning and covering from the original list in [G&J]. The DOMINATING SET problem [GT2] asks, given a graph G and an integer K, whether there is a set of K vertices which, together with their neighborhoods (those vertices adjacent to them), cover all the vertices of the graph. Booth and Johnson [4] have shown that this problem remains NP-complete for *undirected path graphs*, a restricted class of chordal graphs, but can be solved in polynomial time if further restricted to the class of *directed path graphs*. This latter class is a generalization of the class of trees, for which polynomial time algorithms were already known. (Both undirected and directed path graphs are intersection graphs for sets of paths in a tree; the distinction is that in the latter case all edges of the tree are assumed to be directed away from the root, and the set consists only of directed paths).

The HYPERGRAPH 2-COLORABILITY (or SET SPLITTING) problem [SP4] asks, given a set V of *vertices*, and a collection C of *hyperedges* (subsets of V), whether V can be partitioned into two sets in such a way that no hyperedge is entirely contained within either set. This was known to be NP-complete for 3-regular hypergraphs (instances where all hyperedges are of size 3), although it reduces to graph 2-colorability and hence is solvable in polynomial time for 2-regular hypergraphs. It is also solvable in polynomial time for *Steiner triple systems* (3-regular hypergraphs in which each pair or vertices occurs together in exactly one hyperedge), since for these the answer is always ''no.'' However, Phelps [33] has shown that the problem is NP-complete for *partial* Steiner triple systems (3-regular hypergraphs in which each pair of vertices occurs in *at most* one hyperedge). Colbourn *et al.* [12] have shown that analogous results hold for partial and full Steiner quadruple systems (4-regular hypergraphs in which each triple occurs at most/exactly once), although in this case the polynomial time algorithm for full systems is considerably less trivial. If one asks about HYPERGRAPH K-COLORABILITY, $K > 2$, one can get NP-hardness results for more restricted classes of hypergraphs. Phelps and Rödl [34], show that, although 2-colorability is trivial for (full) Steiner triple systems, 11-colorability is NP-complete. Colbourn *et al.* [11] have similar results for other block designs.

With regards to problems from our first two columns, there are only a few brief bits of news. Holyer's NP-completeness proof for CHROMATIC INDEX has at last appeared [19] — this has been one of my most requested ''unpublished manuscripts.'' Progress on the still-open GRAPH ISOMORPHISM problem proceeds, with Luks [27] announcing improved running times for his bounded-valence algorithm, and an $O(c^{\sqrt{n}})$ algorithm for the general case, improving on the previous best exponent of $n^{2/3}$. Finally, Dewdney [14] has independently proved the NP-completeness of the CROSSING NUMBER problem discussed in the previous column, using a transformation from 3SAT.

In the next edition of this column, I plan to discuss problems of permutation, ordering, and routing.

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