Graphs and Combinatorics 5, 315–325 (1989)

Graphs and Combinatorics

Gray Codes for Reflection Groups

J.H. Conway[1], N.J.A. Sloane[2] and Allan R. Wilks[2]

[1] Mathematics Department, Princeton University, Princeton, NJ 08540, USA
[2] Mathematical Sciences Research Center, AT&T Bell Laboratories, Murray Hill, NJ 07974, USA

Abstract. Let G be a finite group generated by reflections. It is shown that the elements of G can be arranged in a cycle (a "Gray code") such that each element is obtained from the previous one by applying one of the generators. The case $G = \mathscr{A}_1^n$ yields a conventional binary Gray code. These generalized Gray codes provide an efficient way to run through the elements of any finite reflection group.

1. Introduction

The classical version of a Gray code is a Hamiltonian circuit through the 2^n vertices of the n-cube, or equivalently an ordering of the 2^n binary vectors of length n such that each pair of adjacent vectors (including the first and last) differ in a single position. For the extensive literature see the bibliography. The first appearance of the "Gray code" that we have located is in 1872 [29].

As we will show, the classical version is the special case $G = \mathscr{A}_1^n$ of the following.

Theorem. *Let G be a finite group generated by reflections $R_1, \ldots, R_n$. Then there is a Hamiltonian circuit in the Cayley diagram for G corresponding to these generators. In other words the $g = |G|$ elements of G can be arranged in order*

$$\{a_0, a_1, \ldots, a_{g-1}\} \tag{1}$$

so that for each i $(0 \le i \le g-1)$ there is a j so that $a_{i+1} = a_i R_j$ (where $a_g = a_0$).

We call (1) a Gray code for G.

It is well-known that any group generated by reflections can be described by a Coxeter diagram [7, 14, 15, 31]. The finite reflection groups for which the Coxeter diagram is a connected graph are ([7], p. 193, Theorem 1) the groups $\mathscr{A}_n$ $(n \ge 1)$, $\mathscr{B}_n$ $(n \ge 2)$, $\mathscr{D}_n$ $(n \ge 4)$, $\mathscr{E}_6$, $\mathscr{E}_7$, $\mathscr{E}_8$, $\mathscr{F}_4$, $\mathscr{G}_2$, $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{I}_2(m)$, $(m = 5$ or $m > 7)$.* These are the *irreducible* reflection groups. Figure 1 shows their Coxeter diagrams,

* We follow Grove and Benson [31] in using script letters for these groups, to distinguish them from the Lie groups and Euclidean lattices with similar names.